DUALITY, TANGENTIAL INTERPOLATION, AND TÖPLITZ CORONA PROBLEMS.

MRINAL RAGHUPATHI† AND BRETT D. WICK‡

Abstract. In this paper we extend a method of Arveson [6] and McCullough [12] to prove a tangential interpolation theorem for subalgebras of H^∞. This tangential interpolation result implies a Töplitz corona theorem. In particular, it is shown that the set of matrix positivity conditions is indexed by cyclic subspaces, which is analogous to the results obtained for the ball and the polydisk algebra by Trent-Wick [16] and Douglas-Sarkar [10].

1. Introduction

The classical corona problem asks whether the set of point evaluations, for points in the unit disk $\mathbb{D}$, is dense in the maximal ideal space of H^∞. A famous result of Carleson [9] shows that they are dense. Let $\mathcal{A}$ be an abelian Banach algebra, and let M be its maximal ideal space. A subset X of M is dense in M if and only if for any finite set of functions $f_1, \ldots, f_n$ such that $\sum_{j=1}^n |f_j(x)|^2 \geq \delta^2 > 0$ for $x \in X$, there exists a set $g_1, \ldots, g_n \in \mathcal{A}$ such that $f_1 g_1 + \cdots + f_n g_n = 1$.

Arveson, [6], studied a related problem replacing the condition $\sum_{j=1}^n |f_j(x)|^2 \geq \delta^2$, by the operator theoretic condition $\sum_{j=1}^n T_{f_j} T_{f_j}^* \geq \delta^2 I$, where T_f is the Töplitz operator with symbol f acting on the Hardy space H^2. He showed that under this assumption there exists $g_1, \ldots, g_n \in H^\infty$ such that $\sum_{j=1}^n f_j g_j = 1$ and $\sum_{j=1}^n |g_j(z)|^2 \leq \delta^{-2}$. The constant δ^{-2} is optimal, as demonstrated by the choice $f_1 = 1$. See Schubert for the best possible constant [15].

In general determining the best constants in the corona problem is considerably challenging. For the Töplitz corona problem we do obtain the optimal constant. However, we make stronger assumptions.

The objective of this paper is to show that a similar Töplitz corona theorem holds for the case where $\mathcal{A}$ is weak*-closed subalgebra of H^∞. Our result makes use of a modification of the Arveson distance formula [6, Theorem 1], a refinement of this due to McCullough [12, Theorem 1]. These modifications allow us to then demonstrate the first main result of this paper, a tangential interpolation theorem for unital weak*-closed algebras $\mathcal{A}$.

Date: March 24, 2010.

2000 *Mathematics Subject Classification.* Primary 47A57; Secondary 30H30, 30H50.

Key words and phrases. Nevanlinna-Pick interpolation, Toeplitz Corona Problem, Distance Formulae, Hilbert Module.

† The first author was supported in part by a National Science Foundation Young Investigator Award, Workshop in Analysis and Probability, Texas A&M University.

‡ The second author is supported by National Science Foundation CAREER Award DMS# 0955432 and an Alexander von Humboldt Fellowship.

1.1. **Notation.** We denote by L^p the Lebesgue space on the unit circle with respect to normalized arc-length measure. The corresponding Hardy space will be denoted H^p.

Given a subset $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, we denote by $[\mathcal{S}]$ the smallest closed subspace that contains $\mathcal{S}$.

A function $g \in H^2$ is called outer if the closure of $H^\infty g$ is H^2. In this paper we adopt the following notation. Given a subalgebra $\mathcal{A} \subseteq H^\infty$ and an outer function g we let K_g be the reproducing kernel of the subspace $[\mathcal{A}g]$, viewed as a subspace of H^2.

1.2. **Statement of main results.** This paper is concerned primarily with tangential interpolation theorems and their application to Töplitz corona problems. We would like to give an overview of the main results. We begin by stating the tangential interpolation problem and our main result, which is Theorem 1.1. In Section 2 we will elaborate on the connections between the two problems.

Let $\mathcal{A}$ be a unital, weak*-closed subalgebra of H^∞. Let $(x_1, \dots, x_n)$ be a sequence of points in the unit disk $\mathbb{D}$, let $(v_1, \dots, v_n)$ be a sequence of vectors in ℓ^2 and let $(w_1, \dots, w_n)$ be a sequence of scalars. We identify a function $F : \mathbb{D} \to \ell^2$ with a sequence of functions $(f_k)_{k=1}^\infty$ in the usual way. Let $F : \mathbb{D} \to \ell^2$ be such that $f_k \in \mathcal{A}$ for all $k \geq 1$. The function F induces an operator $M_F : H^2 \to H^2 \otimes \ell^2$ given by $M_F(h) = Fh$. Hence, M_F can be identified with the column operator $(T_{f_1}, \dots)^t$. Similarly, there is a map from $H^2 \otimes \ell^2 \to H^2$ given by $M_F(h_k) = \sum_{k=1}^\infty f_k h_k$. In this case, the operator M_F is identified with the row operator $(T_{f_1}, \dots)$. We denote by $C(\mathcal{A})$ the set of F such that $f_k \in \mathcal{A}$, for $k \geq 1$, viewed as column operators. When viewed as row operators we use the notation $R(\mathcal{A})$. In both instances the norm of M_F, as an operator, coincides with $\sup_{z\in\mathbb{D}} \|F(z)\|_{\ell^2}$.

We are concerned with the following extremal problem

$$\inf \left\{ \sup_{z\in\mathbb{D}} \|F(z)\|_{\ell^2} : \langle v_j, F(x_j) \rangle = \overline{w_j} \text{ for } j = 1, \dots, n \right\}.$$

We say that a function F such that $\langle v_j, F(x_j) \rangle = \overline{w_j}$ for $j = 1, \dots, n$ is a solution to the tangential interpolation problem.

Our main result is a characterization, in terms of matrix positivity conditions, for the existence of a solution $F \in C(\mathcal{A})$ such that $\sup_{z\in\mathbb{D}} \|F(z)\|_{\ell^2} \leq \alpha$, where α is a prescribed constant.

Theorem 1.1. *Let $\mathcal{A}$ be a unital weak*-closed subalgebra of H^∞. Let $(x_1, \dots, x_n)$ be a sequence of points from the unit disk $\mathbb{D}$, let $(v_1, \dots, v_n)$ be a sequence of ℓ^2 vectors, and let $(w_1, \dots, w_n)$ be a sequence of scalars. Let Q_g denote the matrix*

$$Q_g = \left[(\alpha^2 \langle v_j, v_i \rangle - w_i \overline{w_j}) K_g(x_i, x_j) \right]. \tag{1}$$

Then there exists a function $F : \mathbb{D} \to \ell^2$ such that $\sup_{z\in\mathbb{D}} \|F(z)\|_{\ell^2} \leq \alpha$ and $\langle v_j, F(x_j) \rangle = \overline{w_j}$ if and only if $Q_g \geq 0$ for all outer functions g such that $\|g\|_2 = 1$.

A more careful examination of the proof of Theorem 1.1, which will be given in Section 4, shows that we need only consider a subset of the set of all outer functions.

Before we state this corollary we introduce some notation. Given $\mathcal{A}$ we let $L^\infty(\mathcal{A})$ denote the smallest weak*-closed subalgebra of L^∞ that contains $\mathcal{A} + \overline{\mathcal{A}}$. The algebra $L^\infty(\mathcal{A})$ is the algebra of essentially-bounded measurable functions for some sub-sigma-algebra of the Lebesgue measurable sets on the circle. Therefore,

there exists a sigma-algebra $\mathcal{M}$ consisting of Lebesgue measurable subsets of $\mathbb{T}$ such that $L^\infty(\mathcal{A}) = L^\infty(\mathbb{T}, \mathcal{M}, dm)$, where m is Lebesgue measure. We let $L^p(\mathcal{A})$ denote the corresponding L^p space.

If g is an outer function, then we denote $\mathcal{H}_g = [\mathcal{A}g]$. Recall that the kernel function for this space is K_g. When $g = 1$ we denote $\mathcal{H}_g$ by $\mathcal{H}$ and the corresponding kernel is denoted K.

Corollary 1.2. *Retaining the notation of Theorem 1.1.*

1. *There exists a function* $F \in C(\mathcal{A})$ *such that* $\sup_{z\in\mathbb{D}} \|F(z)\|_{\ell^2} \leq \alpha$ *and* $\langle v_j, F(x_j)\rangle = \overline{w_j}$ *for* $j = 1, \dots, n$ *if and only if* $Q_g \geq 0$ *for all outer functions* g *such that* $|g| \in L^2(\mathcal{A})$ *and* $\|g\|_2 = 1$.
2. *For* $F \in R(\mathcal{A})$ *let* $M_{F,g} \in B(\mathcal{H}_g \otimes \ell^2, \mathcal{H}_g)$ *be given by* $h \to Fh$. *If there is a constant* $\delta > 0$ *such that* $M_{F,g}M_{F,g}^* \geq \delta^2 I$ *for all outer functions* g *such that* $|g| \in L^2(\mathcal{A})$ *and* $\|g\|_2 = 1$, *then there exists a function* G *in* $C(\mathcal{A})$ *such that* $FG = 1$ *and* $\sup_{z\in\mathbb{D}} \|G(z)\| \leq \delta^{-1}$.

The difficulty with Theorem 1.1 is that the positivity condition is over a whole family of outer functions or kernels. In some applications we would rather have the condition over just a single kernel. We turn to establishing a tangential interpolation theorem where we replace the family of conditions $Q_g \geq 0$ for all outer functions g such that $|g| \in L^2(\mathcal{A})$ by a single positivity condition. However, we can not guarantee a solution of optimal norm. This leads to the second main result of the paper.

Theorem 1.3. *Let* $\mathcal{A}$ *be a unital weak***-closed subalgebra of* H^∞, *and let* $\mathcal{H}_g$ *be the subspace generated by* $\mathcal{A}g$, *where* g *is an outer function. Suppose that for each outer function* g *such that* $|g| \in L^2(\mathcal{A})$ *there exists a similarity* $S_g : \mathcal{H} \to \mathcal{H}_g$. *Also assume that there is a constant* c, *that is independent of* g, *such that* $\|S_g\|\,\|S_g^{-1}\| \leq c$ *for all such outer functions* g.

1. *If* $[(\alpha^2 \langle v_j, v_i\rangle - w_i\overline{w_j})K(x_i, x_j)] \geq 0$, *then there exists* $F \in C(\mathcal{A})$ *such that* $\sup_{z\in\mathbb{D}} \|F(z)\|_{\ell^2} \leq \alpha c$ *and* $\langle v_j, F(x_j)\rangle = \overline{w_j}$ *for* $j = 1, \dots, n$.
2. *If* $M_F M_F^* \geq \delta^2$ *on* $B(\mathcal{H})$, *then there exists* $G \in C(\mathcal{A})$ *such that* $FG = 1$, *and* $\|G\|_{C(\mathcal{A})} \leq c\delta^{-1}$.

The outline of the paper is as follows. In Section 2 we give background to the tangential interpolation problem, including standard background for reproducing kernel spaces. In Section 3 we provide an extension of Arveson's Distance formula needed in our context. Section 4 puts the computations and ideas from the first two sections together to prove Theorem 1.1. Finally, in Section 5 we prove Theorem 1.3, which essentially follows from Theorem 1.1, and then collect applications of Theorem 1.3 to the case of bounded analytic functions on Riemann surfaces. This application generalizes a result of Ball [7].

The corona problem and its variant the Töplitz corona problem have been studied extensively in the past. The paper of Agler-McCarthy [3, Section 7] provides an excellent overview of the connection between matrix positivity conditions, families of kernels and corona problems. The connections between interpolation theory and Toëplitz corona problem for the bidisk and the Schur-Agler class are described in Agler-McCarthy [2] and Ball-Trent [8].

2. The tangential interpolation problem

In order to state our results and describe our setting we will need the terminology of reproducing kernel Hilbert spaces. We begin with a brief description. The reader should consult the text of Agler and McCarthy [1], or the paper of Aronszajn [5].

2.1. **Reproducing kernel Hilbert spaces.** Let X be a set and let $\mathcal{C}$ be a Hilbert space. We denote by $\mathcal{F}(X,\mathcal{C})$ the set of functions from X to $\mathcal{C}$. A subset $\mathcal{H} \subseteq \mathcal{F}(X,\mathcal{C})$ is called a $\mathcal{C}$-valued reproducing kernel Hilbert space (RKHS) if $\mathcal{H}$ is a Hilbert space and for each $x \in X$, the evaluation map $E_x : \mathcal{H} \to \mathcal{C}$ defined by $f \mapsto f(x)$ is a bounded linear map on $\mathcal{H}$. The kernel function of $\mathcal{H}$ is the map $K : X \times X \to B(\mathcal{C})$ defined by $K(x,y) = E_x E_y^* \in B(\mathcal{C})$. It is straightforward that the kernel function of $\mathcal{H}$ is a positive semidefinite function on $X \times X$ and that the span of $\{E_x\xi : x \in X, \xi \in \mathcal{C}\}$ is dense in $\mathcal{H}$.

Conversely, every $B(\mathcal{C})$-valued positive semidefinite function K on $X \times X$ gives rise to a $\mathcal{C}$-valued RKHS $\mathcal{H}(K)$ in a canonical way and this correspondence is one-to-one [1].

We denote the Hilbert space associated to K by $\mathcal{H}(K)$. We suppress the kernel function, when the context is clear.

For $i = 1, 2$, let K_i be a $\mathcal{C}_i$-valued kernel function on X and let $\mathcal{H}_i = \mathcal{H}(K_i)$. Given a function $F : X \to \mathcal{F}(\mathcal{C}_1, \mathcal{C}_2)$ and a function $g : X \to \mathcal{C}_1$, let Fg denote the pointwise product of F and g. We say that $F : X \to \mathcal{F}(\mathcal{C}_1, \mathcal{C}_2)$ is a multiplier from $\mathcal{H}_1$ to $\mathcal{H}_2$ if and only if $Fg \in \mathcal{H}_2$ for all $g \in \mathcal{H}_1$. We denote the set of multipliers from $\mathcal{H}_1$ to $\mathcal{H}_2$ by $\operatorname{mult}(\mathcal{H}_1, \mathcal{H}_2)$. Since the space $\mathcal{H}_i$ is completely determined by its kernel function K_i we also use the notation $\operatorname{mult}(K_1, K_2)$ to denote the space of multipliers.

The closed graph theorem shows that the operator $M_F : \mathcal{H}_1 \to \mathcal{H}_2$ defined by $M_F(g) = Fg$ is bounded. The multiplier norm of a function $F \in \operatorname{mult}(K_1, K_2)$ is defined as $\|F\|_{\operatorname{mult}(K_1,K_2)} := \|M_F\|_{B(\mathcal{H}_1,\mathcal{H}_2)}$.

If $E_{i,x}$ denotes the evaluation map on $\mathcal{H}_i$, and $F \in \operatorname{mult}(K_1, K_2)$, then

$$E_{2,x} M_F = F(x) E_{1,x} \text{ for all } x \in X.$$

If $\mathcal{H}$ is a scalar-valued RKHS, then the evaluation map E_x is a linear functional and the unique element $k_x \in \mathcal{H}$ such that $f(x) = E_x(f) = \langle f, k_x \rangle$ for all $f \in \mathcal{H}$ is called the kernel function at the point x for $\mathcal{H}$. In this case $K(x,y) = E_x E_y^* = \langle k_y, k_x \rangle$.

Given two scalar-valued RKHSs $\mathcal{H}_i$, for $i = 1, 2$, with kernel functions K_i, and $f \in \operatorname{mult}(\mathcal{H}_1, \mathcal{H}_2)$, we have $M_f^* k_{2,x} = \overline{f(x)} k_{1,x}$, where $k_{i,x}$ denotes the kernel function for $\mathcal{H}_i$ at the point x.

Given a scalar-valued RKHS $\mathcal{H}(K)$ we give the Hilbert space tensor product $\mathcal{H} \otimes \mathcal{C}$ the structure of a $\mathcal{C}$-valued RKHS by defining $E_x(f \otimes \xi) = f(x)\xi$. A short calculation reveals that the kernel function for $\mathcal{H} \otimes \mathcal{C}$ is $K(x,y) I_{\mathcal{C}}$, where $I_{\mathcal{C}}$ is the identity map on $\mathcal{C}$, and that $E_x^* \xi = k_x \otimes \xi$.

If $F \in \operatorname{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})$, then, for each $x \in X$, $F(x) \in B(\mathbb{C}, \mathcal{C})$. We identify $B(\mathbb{C}, \mathcal{C})$ with $\mathcal{C}$ via the correspondence $T \mapsto T(1)$. We have,

$$\langle M_F^*(k_x \otimes \xi), h \rangle = \langle k_x \otimes \xi, Fh \rangle = \langle k_x, h \rangle F(x)^* \xi = \langle \langle \xi, F(x) \rangle k_x, h \rangle .$$

Therefore,

$$M_F^*(k_x \otimes \xi) = (F(x)^* \xi) k_x = \langle \xi, F(x) \rangle k_x. \tag{2}$$

Given $F \in \operatorname{mult}(\mathcal{H} \otimes \mathcal{C}, \mathcal{H})$ we have $F(x) \in B(\mathcal{C}, \mathbb{C})$ and so $F(x)^* \in B(\mathbb{C}, \mathcal{C}) = \mathcal{C}$, under our identification. In this case we have,

$$\begin{aligned} \langle h \otimes \xi, M_F^* k_x \rangle &= \langle F(h \otimes \xi), k_x \rangle \\ &= h(x) F(x) \xi = \langle h, k_x \rangle \langle \xi, F(x)^* \rangle \\ &= \langle h \otimes \xi, k_x \otimes F(x)^* \rangle . \end{aligned}$$

Hence,

$$M_F^* k_x = k_x \otimes F(x)^*. \tag{3}$$

In this paper we will be interested primarily in two special cases: the case where $\mathcal{C}_1 = \mathcal{C}_2 = \mathbb{C}$, and the case where either $\mathcal{C}_1$ or $\mathcal{C}_2$ is $\ell^2 := \ell^2(\mathbb{N})$ and the other is $\mathbb{C}$.

We can view an operator $T \in B(\mathcal{H}, \mathcal{H} \otimes \ell^2)$ as a column operator matrix of the form $T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \end{bmatrix}$. If $\mathcal{A} \subseteq B(\mathcal{H})$, then we denote by $C(\mathcal{A})$ the set of column operators $[T_i]$ such that $T_i \in \mathcal{A}$ for all i. There is a similar identification of $B(\mathcal{H} \otimes \ell^2, \mathcal{H})$ with the set of row operator matrices, and we denote the set of row operators with entries from $\mathcal{A}$ by $R(\mathcal{A})$. It is easily checked that $\operatorname{mult}(\mathcal{H}, \mathcal{H} \otimes \ell^2) = C(\operatorname{mult}(\mathcal{H}))$ and that $\operatorname{mult}(\mathcal{H} \otimes \ell^2, \mathcal{H}) = R(\operatorname{mult}(\mathcal{H}))$.

2.2. **Tangential interpolation.** We now describe the tangential interpolation problem. Let X be a set and let K be a kernel on X. We will assume that $K(x, x) \neq 0$, for all $x \in X$.

Given a finite sequence of points $(x_1, \ldots, x_n) \in X^n$, a sequence of vectors $(v_1, \ldots, v_n) \in \mathcal{C}^n$, and a sequence of scalars $(w_1, \ldots, w_n)$. We say that a function $F \in \operatorname{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})$ is a solution to the associated tangential interpolation problem if and only if $F(x_j)^* v_j = \overline{w_j}$ for $j = 1, \ldots, n$.

Given a constant α we are interested in finding necessary and sufficient conditions for the existence of a solution of norm at most α, that is, a multiplier F such that $\|F\|_{\operatorname{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})} \leq \alpha$ and

$$F(x_j)^* v_j = \overline{w_j} \text{ for } j = 1, \ldots, n.$$

As is the case with many complex interpolation problems of this type, there is a necessary matrix positivity condition, which we now derive. Let F be a solution to the above problem and let $\alpha \geq \|F\|_{\operatorname{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})}$. For $x, y \in X$ and $\xi, \zeta \in \mathcal{C}$ we have,

$$\begin{aligned} \langle M_F M_F^* (k_y \otimes \zeta), k_x \otimes \xi \rangle &= \langle M_F^* (k_y \otimes \zeta), M_F^* (k_x \otimes \xi) \rangle \\ &= \langle (F(y)^* \zeta) k_y, (F(x)^* \xi) k_x \rangle \\ &= (F(y)^* \zeta)(\overline{F(x)^* \xi}) K(x, y) \end{aligned} \tag{4}$$

Consider the restriction of M_F^* to the subspace $\mathcal{K}$, that is the span of the vectors $\{k_{x_1} \otimes v_1, \ldots, k_{x_n} \otimes v_n\}$. An element of $\mathcal{K}$ is of the form $k = \sum_{j=1}^n c_j k_{x_j} \otimes v_j$. Since M_F^* has norm at most α we see that $\left\langle (\alpha^2 I - M_F M_F^*) k, k \right\rangle \geq 0$. Substituting for k, using (4) and the fact that $F(x_j)^* v_j = \overline{w_j}$, shows us that $\|M_F^*|_{\mathcal{K}}\| \leq \alpha$ if and only if

$$[(\alpha^2 \langle v_j, v_i \rangle - w_i \overline{w_j}) K(x_i, x_j)] \geq 0. \tag{5}$$

If $\mathcal{C} = \mathbb{C}$, and $v_i = 1 \in \mathbb{C}$, then the above positivity condition is a necessary condition for the existence of a function $f \in \operatorname{mult}(K)$ such that $\|f\|_{\operatorname{mult}(K)} \leq \alpha$

and $f(x_j) = w_j$ for $j = 1, \ldots, n$. If $K(z, w) = (1 - z\overline{w})^{-1}$ is the Szegö kernel for the unit disk $\mathbb{D}$, then the associated multiplier algebra is $H^\infty(\mathbb{D})$ and the multiplier norm is the supremum norm. This is the setting of the classical Nevanlinna-Pick theorem. In this case, it is a well-known fact that the necessary matrix positivity condition is also sufficient.

In general, a single matrix positivity condition is not sufficient to guarantee that there exist solutions of norm at most a given constant α. However, in certain situations we may be able to replace a single kernel function by a set of kernel functions $\{K_\lambda : \lambda \in \Lambda\}$ such that $\mathrm{mult}(K_\lambda) = \mathrm{mult}(\mathcal{H})$. In addition, this collection of kernel functions has the property that the condition $Q_\lambda = [(\alpha^2 - w_i\overline{w_j})K_\lambda(x_i x_j)] \geq 0$ for all $\lambda \in \Lambda$ is equivalent to the existence of a multiplier $f \in \mathrm{mult}(\mathcal{H})$ such that $\|f\|_{\mathrm{mult}(\mathcal{H})} \leq \alpha$ and $f(x_j) = w_j$. This is, in fact, the situation when we replace the algebra $H^\infty(\mathbb{D})$ by a weak*-closed subalgebra $\mathcal{A}$ of $H^\infty(\mathbb{D})$ [13].

To make this formal, we introduce the following definition.

Definition 2.1. Let $\mathcal{H}$ be a scalar-valued RKHS on a set X. A set of kernels $\{K_\lambda : \lambda \in \Lambda\}$ on X such that $\mathrm{mult}(K_\lambda) \supseteq \mathrm{mult}(\mathcal{H})$ has the *tangential interpolation property* for $\mathrm{mult}(\mathcal{H})$ if and only if for every finite sequence of points $(x_1, \ldots, x_n)$ from X, vectors $(v_1, \ldots, v_n)$ from $\mathcal{C}$, and scalars $(w_1, \ldots, w_n)$ the condition

$$Q_\lambda := [(\alpha^2 \langle v_j, v_i \rangle - w_i\overline{w_j})K_\lambda(x_i, x_j)]_{i,j=1}^n \geq 0 \text{ for all } \lambda \in \Lambda \tag{6}$$

implies the existence of a multiplier $F \in \mathrm{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})$ such that

$$\|F\|_{\mathrm{mult}(\mathcal{H}, \mathcal{H}\otimes\mathcal{C})} \leq \alpha$$

and $F(x_j)^* v_j = \overline{w_j}$ for $j = 1, \ldots, n$.

2.3. **Reformulation as a distance problem.** Suppose that we are given the data of tangential interpolation problem, that is, a sequence of vectors $(v_1, \ldots, v_n)$ in ℓ^2, a sequence of points $(x_1, \ldots, x_n)$ in X and a sequence of scalars $(w_1, \ldots, w_n)$. Let us assume that there are solutions to this problem, that is, multipliers $F \in \mathrm{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})$ such that $F(x_j)^* v_j = \overline{w_j}$ for $j = 1, \ldots, n$. Let $\mathcal{J}$ denote the set of functions G in $\mathrm{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})$ such that $G(x_j)^* v_j = 0$ for $j = 1, \ldots, n$. Given two solutions F_1, F_2 to the tangential interpolation problem we see that the difference $F_1 - F_2 \in \mathcal{J}$. Conversely, every solution must lie in $F + \mathcal{J}$. Hence, the set of solutions to the interpolation problem is precisely $F + \mathcal{J}$, where F is one particular solution.

We will show in Section 4 that under the assumption $K(x, x) \neq 0$ for all $x \in X$, the multiplier algebra $\mathrm{mult}(K)$ is a unital, weak*-closed subalgebra of $B(\mathcal{H}(K))$. We will also establish the fact that the evaluation map on $\mathrm{mult}(K)$, given by $f \mapsto f(x)$, is weak*-continuous.

Since the algebras we are interested in are weak*-closed the least norm of any solution to the interpolation problem is given by $\inf\{\|F + G\|_{\mathrm{mult}(\mathcal{H}, \mathcal{H}\otimes\mathcal{C})} : G \in \mathcal{J}\}$. This is the distance from F to $\mathcal{J}$, that is, $d(F, \mathcal{J})$. The problem of determining necessary and sufficient conditions for the existence of a solution of norm at most α is reduced to the problem of computing a formula for the distance $d(F, \mathcal{J})$. With this in mind we present a refinement of a distance formula in [6, 12] in the next section.

We will show how to deduce a Töplitz corona theorem from a tangential interpolation theorem in Section 4.

3. A Distance Formula

In this section we recall some basic facts about the distance of an operator $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ from a weak*-closed subspace. In the next section we will use this formula to compute the distance of a solution to the subspace $\mathcal{J}$ and thus obtain a tangential interpolation theorem. This a simple application of standard duality techniques.

We begin by recalling some basic facts related to the dual Banach space structure of $B(\mathcal{H})$. Let $\mathcal{H}$ be a separable Hilbert space. Given an operator $T \in B(\mathcal{H})$, the trace of T is defined by

$$\mathrm{trace}(T) = \sum_{j=1}^{\infty} \langle Te_j, e_j \rangle$$

where $\{e_j : j \in \mathbb{N}\}$ is an orthonormal basis. The sum in the definition of the trace does not depend on the choice of orthonormal basis. An operator is called a trace class operator if and only if trace(T) is finite. The set of trace class operators is an ideal in $B(\mathcal{H})$ and is a Banach space in the trace class norm $\|T\|_1 = \mathrm{trace}(|T|)$. We let $TC(\mathcal{H})$ denote the space of trace class operators on $\mathcal{H}$. It is well known that the dual of $TC(\mathcal{H})$ can be identified naturally with $B(\mathcal{H})$ and that the dual pairing is given by

$$\langle T, A \rangle = \mathrm{trace}(TA), \text{ where } A \in B(\mathcal{H}), T \in TC(\mathcal{H}).$$

This pairing also identifies $TC(\mathcal{H})$ with the set of weak*-continuous linear functionals on $B(\mathcal{H})$. An operator $H \in B(\mathcal{H})$ is called a Hilbert-Schmidt operator if and only if $HH^* \in TC(\mathcal{H})$, that is, $\sum_{j=1}^{\infty} \langle He_j, He_j \rangle$ is finite. The set of all Hilbert-Schmidt operators will be denoted $HS(\mathcal{H})$. This space is a Hilbert space when given the inner product $\langle H, K \rangle = \mathrm{trace}(HK^*)$. Given a trace class operator T there exist Hilbert-Schmidt operators H, K such that $T = HK^*$ and $\|T\|_1^{1/2} = \|H\|_2 = \|K\|_2$. Let $A \in B(\mathcal{H})$, let $T = HK^* \in TC(\mathcal{H})$, let $h_j = He_j$ and let $k_j = Ke_j$. If $h = \oplus h_j$ and $k = \oplus k_j \in \mathcal{H} \otimes \ell^2$, then,

$$\begin{aligned} \mathrm{trace}(AT) &= \mathrm{trace}(AHK^*) = \mathrm{trace}(K^*AH) \\ &= \sum_{j=1}^{\infty} \langle AHe_j, Ke_j \rangle = \sum_{j=1}^{\infty} \langle Ah_j, k_j \rangle = \langle (A \otimes I)h, k \rangle . \end{aligned}$$

We also have

$$\|h\|^2 = \sum_{j=1}^{\infty} \|h_j\|^2 = \sum_{j=1}^{\infty} \langle He_j, He_j \rangle = \mathrm{trace}(H^*H) = \|H\|_2^2 .$$

Hence, $\|h\| = \|H\|_2$ and $\|k\| = \|K\|_2$.

Given an operator $A \in B(\mathcal{H})$ and a weak*-closed subspace $\mathcal{S} \subseteq B(H)$, the distance from A to $\mathcal{S}$ is given by

$$d(A, \mathcal{S}) = \inf\{\|A + S\| : S \in \mathcal{S}\} = \sup\{|\mathrm{trace}(AT)| : T \in \mathcal{S}_\perp, \|T\|_1 = 1\}$$

where $\mathcal{S}_\perp$ denotes the preannihilator of $\mathcal{S}$. If we write $T = HK^*$ where $H, K \in HS(\mathcal{H})$ and $\|T\|_1^{1/2} = \|H\|_2 = \|K\|_2$, $h_j = He_j$, and $k_j = Ke_j$ as before, then $\|h\| = \|k\| = 1$. Since $T = HK^* \in \mathcal{S}_\perp$ we have

$$0 = \mathrm{trace}(SHK^*) = \sum_{j=1}^{\infty} \langle Sh_j, k_j \rangle = \langle (S \otimes I)h, k \rangle ,$$

for all $S \in \mathcal{S}$. Hence, $k \perp (\mathcal{S} \otimes I)h$. Rewriting the distance formula we get,

$$d(A, \mathcal{S}) = \sup\{|\langle (A \otimes I)h, k\rangle| : \|h\| = \|k\| = 1, k \perp (\mathcal{S} \otimes I)h\}. \tag{7}$$

It will also prove useful to have such a formula when $\mathcal{S} \subseteq B(\mathcal{H}_1, \mathcal{H}_2)$. Let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. We can identify $B(\mathcal{H}_1, \mathcal{H}_2)$ with a subspace of $B(\mathcal{H})$ in the usual way, that is, $A(h_1 \oplus h_2) = 0 \oplus Ah_1$. Keeping the notation from (7) we can write $h \in (\mathcal{H}_1 \oplus \mathcal{H}_2) \otimes \ell^2$ as $h = h_1 \oplus h_2$, where $h_i \in \mathcal{H}_i \otimes \ell^2$ for $i = 1, 2$. We have,

$$\langle (A \otimes I)h, k\rangle = \langle 0 \oplus (A \otimes I)h_1, k_1 \oplus k_2\rangle = \langle (A \otimes I)h_1, k_2\rangle .$$

Since $k \perp (\mathcal{S} \otimes I)h$ we see that $k_2 \perp (\mathcal{S} \otimes I)h_1$. Hence,

$$d(A, \mathcal{S}) = \sup\{|\langle (A \otimes I)h_1, k_2\rangle| : \|h_1\| = 1, \|k_2\| = 1, k_2 \perp (\mathcal{S} \otimes I)h_1\}.$$

These computations prove the following distance formula:

Theorem 3.1. *Let $\mathcal{S}$ be a weak*-closed subspace of $B(\mathcal{H}_1, \mathcal{H}_2)$ and let $A \in B(\mathcal{H}_1, \mathcal{H}_2)$. The distance of A from $\mathcal{S}$ is given by*

$$d(A, \mathcal{S}) = \sup\{|\langle (A \otimes I)h_1, h_2\rangle| : h_i \in \mathcal{H}_i \otimes \ell^2, \|h_i\| = 1, h_2 \perp (\mathcal{S} \otimes I)h_1\}. \tag{8}$$

4. A Tangential interpolation theorem for subalgebras of H^∞

4.1. **Töplitz corona problem.** We had made the claim at the end of Section 2 that the multiplier algebra was weak*-closed and that point evaluations are weak*-continuous. We now prove that claim.

Lemma 4.1. *Let K be a kernel on a set X such that $K(x, x) \neq 0$ for all $x \in X$. Then the algebra $\operatorname{mult}(\mathcal{H})$ is weak*-closed when viewed as a subalgebra of $B(\mathcal{H}(K))$. In addition, the evaluation map on $\operatorname{mult}(K)$ given by $f \mapsto f(x)$ is weak*-continous.*

Proof. Let M_{f_t} be a net that converges to an operator $T \in B(\mathcal{H})$ in the weak*-topology. We have, $\langle M_{f_t}h, k\rangle \to \langle Th, k\rangle$ for any pair of vectors $h, k \in \mathcal{H}$. Let $k = k_x$ and $h = k_y$. We have,

$$\langle Tk_y, k_x\rangle = \lim_t \langle M_{f_t}k_y, k_x\rangle = \lim_t \left\langle k_y, M_{f_t}^* k_x\right\rangle = \lim_t f_t(x) \langle k_y, k_x\rangle .$$

If we choose $x = y$, and use the fact that $K(x, x) \neq 0$, then $\lim_t f_t(x) = \frac{\langle Tk_x, k_x\rangle}{K(x,x)}$. Hence, $\lim_t f_t(x)$ exists. Let us denote the limit by $f(x)$. We have $\langle h, T^*k_x\rangle = f(x) \langle h, k_x\rangle$ for all $h \in \mathcal{H}$ and $x \in X$, and so $T^*k_x = \overline{f(x)}k_x$. It follows that f is a multiplier of $\mathcal{H}$ and that $T = M_f$.

The above argument also shows that if $M_{f_t} \to M_f$ in the weak*-topology, then $f_t(x) \to f(x)$. □

We now show how the hypothesis of the Töplitz corona problem leads to a tangential interpolation problem.

Suppose that $F \in \operatorname{mult}(\mathcal{H} \otimes \mathcal{C}, \mathcal{H})$ and that $M_F M_F^* \geq \delta^2 I$. We have, $M_F^* k_x = k_x \otimes F(x)^*$, where we have identified $B(\mathcal{C}, \mathbb{C})$ with $\mathcal{C}$. Hence,

$$\begin{aligned} \langle M_F^* k_y, M_F^* k_x\rangle &= \langle k_y \otimes F(y)^*, k_x \otimes F(x)^*\rangle \\ &= \langle F(y)^*, F(x)^*\rangle K(x, y) \end{aligned}$$

Let $Y \subseteq X$ and let $\mathcal{K}_Y$ be the span of $\{k_x : x \in Y\}$. Since the space $\mathcal{H}$ is the closed linear span of the set of kernel functions k_x for $x \in X$, the operator $M_F M_F^* - \delta^2 I$ is

positive if and only if $(M_F M_F^* - \delta^2 I)|_{\mathcal{K}_Y}$ is positive for all finite sets $Y \subseteq X$. The latter condition is equivalent to the positivity of the matrix

$$[(\langle F(y)^*, F(x)^* \rangle - \delta^2)K(x,y)]_{x,y \in Y} \tag{9}$$

for all finite sets $Y \subseteq X$.

Proposition 4.2. *Let $\mathcal{H}$ be an RKHS on X and let $\{K_\lambda : \lambda \in \Lambda\}$ be a set of kernels with the tangential interpolation property for* $\mathrm{mult}(\mathcal{H})$*. Let $\mathcal{H}_\lambda = \mathcal{H}(K_\lambda)$, let $F \in \mathrm{mult}(\mathcal{H} \otimes \mathcal{C}, \mathcal{H})$, and let $M_{F,\lambda}$ denote the operator of multiplication by F between the spaces $\mathcal{H}_\lambda \otimes \mathcal{C}$ and $\mathcal{H}_\lambda$. Suppose that for each $\lambda \in \Lambda$, we have $M_{F,\lambda} M_{F,\lambda}^* \geq \delta^2 I_\lambda$. Then there exists a multiplier $G \in \mathrm{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})$ such that $\|G\| \leq \delta^{-1}$ and $FG = 1$.*

Proof. Since $M_{F,\lambda} M_{F,\lambda}^* \geq \delta^2 I_\lambda$, given a finite set $Y = \{x_1, \ldots, x_n\}$, (9) shows that the matrix $[(\langle F(x_j)^*, F(x_i)^* \rangle - \delta^2)K_\lambda(x_i, x_j)] \geq 0$ for all $\lambda \in \Lambda$. This matrix is of the form Q_λ in (6) for the case where the vectors are $v_j = F(x_j)^*$, and the scalars $w_j = \delta$ for $j = 1, \ldots, n$. Since, K_λ has the tangential interpolation property, there is a contractive multiplier $G_Y \in \mathrm{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})$ such that $\delta = G_Y(x_j)^* F(x_j)^* = (F(x_j)G_Y(x_j))^*$ for $j = 1, \ldots, n$. Since $\delta > 0$ we get, $F(x_j)G_Y(x_j) = \delta$ for $j = 1, \ldots, n$.

The net G_Y is contained in the unit ball of the space $\mathrm{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})$, which is weak*-compact subset of $B(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})$. If $\mathcal{F}$ denotes the collection of all finite subsets of X, then then there is a subnet $\{F_t\} \subseteq \mathcal{F}$ such that of $\{G_{F_t}\}$ converges in the weak*-topology to a contractive multiplier G. Since point evaluations are weak*-continuous on $\mathrm{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})$ we see, for a fixed $x \in X$, that $F(x)G(x) = \lim_{F_t} F(x) G_{F_t}(x) = \delta$. Therefore $FG = \delta$. It follows that $F(\delta^{-1}G) = 1$ and $\|\delta^{-1}G\|_{\mathrm{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})} \leq \delta^{-1}$. □

We have established that if $\{K_\lambda : \lambda \in \Lambda\}$ is a set of kernels that have the tangential interpolation property, then the condition $M_{F,\lambda} M_{F,\lambda}^* \geq \delta^2 I_\lambda$ implies the existence of a multiplier $G \in \mathrm{mult}(\mathcal{H}, \mathcal{H} \otimes \mathcal{C})$ such that $\|G\| \leq \delta^{-1}$ and $FG = 1$.

Our strategy for the proof of Theorem 1.1 is to exploit the distance formula and the existence of at least *one* solution to the tangential interpolation problem.

4.2. **The existence of solutions.** Now let $\mathcal{A}$ be a unital weak*-closed subalgebra of the multiplier algebra of $\mathcal{H}(K)$. Let $g \in \mathcal{H}$ be a nonvanishing function. In particular, recall that an outer function g does not vanish at any point in the disk. We view $\mathrm{mult}(\mathcal{H})$ as a subalgebra of $B(\mathcal{H})$. Let $\mathcal{H}_g$ be the closure of $\mathcal{A}g$ in $\mathcal{H}$, let K_g be the kernel of $\mathcal{H}_g$, let k_x^g be the kernel function at the point x, and let $Q_g = [(\langle v_j, v_i \rangle - w_i \overline{w_j})K_g(x_i, x_j)]$. We have assumed that $\alpha = 1$. However, since this amounts to a rescaling, there is no loss of generality in doing so. Since g does not vanish at any point $x \in X$, the kernel $K_g(x,x) \neq 0$ for any x. Therefore, the results of the previous section do apply.

We will establish the fact the positivity of the matrix Q_g implies the existence of a multiplier $F \in C(\mathcal{A})$ such that $F(x_j)^* v_j = \overline{w_j}$. We will then establish the fact that the closure of $\mathcal{J}g$ in $\mathcal{H}_g \otimes \ell^2$ is the set of functions $f \in \mathcal{H}_g \otimes \ell^2$ such that $\langle f(x_j), v_j \rangle = 0$ for $j = 1, \ldots, n$.

We say that the algebra $\mathcal{A}$ separates x and y if and only if there exists a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. We say that the algebra $\mathcal{A}$ separates a set of points Y if and only if $\mathcal{A}$ separates x and y for all $x, y \in Y$.

Lemma 4.3. *Every element of $\mathcal{A}$ is a multiplier of $\mathcal{H}_g$.*

Proof. Let $h \in \mathcal{H}_g$ and let f_n be a sequence in $\mathcal{A}$ such that $\|f_n g - h\|_{\mathcal{H}} \to 0$. Let $f \in \mathcal{A}$. We have $\|(ff_n)g - fh\|_{\mathcal{H}} \leq \|M_f\| \, \|f_n g - h\|_{\mathcal{H}} \to 0$. Hence, $fh \in \mathcal{H}_g$. $\square$

Lemma 4.4. *The algebra $\mathcal{A}$ separates x and y if and only if k_x^g and k_y^g are linearly independent.*

Proof. Suppose that $\mathcal{A}$ does separate x, y and that $f \in \mathcal{A}$ with $f(x) = 1$ and $f(y) = 0$. Note that f is a multiplier of $\mathcal{H}_g$. Assume that $\alpha k_x^g + \beta k_y^g = 0$. We have

$$0 = M_f^*(\alpha k_x^g + \beta k_y^g) = \alpha \overline{f(x)} k_x^g + \beta \overline{f(y)} k_y^g = \alpha k_x^g.$$

Since g is a nonvanishing function in $\mathcal{H}_g$ we know that $k_x^g \neq 0$ and so $\alpha = 0$. A similar argument shows that $\beta = 0$.

Conversely, suppose that $\mathcal{A}$ does not separate x and y. Let $z \in X$ and let f_n be a sequence in $\mathcal{A}$ such that $f_n g \to k_z^g$. We have $k_z^g(x) = \lim_{n\to\infty} f_n(x) g(x)$. On the other hand, $f_n(x) = f_n(y)$ and so

$$\begin{aligned} k_z^g(y) &= \lim_{n\to\infty} f_n(y) g(y) = \lim_{n\to\infty} f_n(x) g(y) \\ &= \lim_{n\to\infty} f_n(x) g(x) \frac{g(y)}{g(x)} = k_z^g(x) \frac{g(y)}{g(x)}. \end{aligned}$$

Hence, $g(x) K_g(y, z) - g(y) K_g(x, z) = 0$ for all $z \in X$ and so $\overline{g(x)} k_y^g - \overline{g(y)} k_x^g = 0$ with $g(x), g(y) \neq 0$. $\square$

The relation $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in \mathcal{A}$ is an equivalence relation on X. Let $\{x_1, \ldots, x_n\}$ be given. Let us reorder the points $\{x_1, \ldots, x_n\}$ in such a way that there is a sequence $n_0 = 0 < n_1 < \cdots < n_p = n$ such that the sets $X_k = \{x_i : n_{k-1} < i \leq n_k\}$ are the equivalence classes for the above equivalence relation.

Lemma 4.5. *If $Q_g \geq 0$, then there exists a multiplier $F \in C(\mathcal{A})$ such that $F(x_j)^* v_j = \overline{w_j}$ for $j = 1, \ldots, n$. In addition, the subspace $[\mathcal{J} g]$ is precisely the set of functions in $\mathcal{H}_g \otimes \ell^2$ such that $\langle f(x_j), v_j \rangle = 0$ for $j = 1, \ldots, n$*

Proof. By Lemma 4.4 there exist functions $e_1, \ldots, e_p$ such that $e_k|_{X_l}(x) = \delta_{k,l}$ for $k, l = 1, \ldots, p$.

To simplify notation let $K = K_g$, let $Q = [(\langle v_j, v_i \rangle - w_i \overline{w_j}) K(x_i, x_j)]_{i,j=1}^n = [q_{i,j}]$ and let $Q_k = [q_{i,j}]_{n_{k-1} < i,j \leq n_k}$. Let k be given and let $t = n_k - n_{k-1}$. We temporarily set $Y = X_k$ and relabel the points of the set X_k as $y_1, \ldots, y_t$. We also relabel the corresponding vectors as $v_1, \ldots, v_t$ and the scalars $w_1, \ldots, w_t$. Let $e = e_k$.

Since the algebra $\mathcal{A}$ fails to separate any two points of Y we see that there exists a sequence of nonzero scalars $\lambda_1, \ldots, \lambda_t$ such that $k_{y_i} = \lambda_i k_{y_1}$. The matrix Q_k is given by

$$Q_k = [(\langle v_j, v_i \rangle - w_i \overline{w_j}) K(y_1, y_1) \lambda_i \overline{\lambda_j}].$$

Since this is a square submatrix of Q we know that $Q_k \geq 0$. Since the λ_i are nonzero and $K(y_1, y_1)$ is nonzero we see that $[\langle v_j, v_i \rangle] \geq [w_i \overline{w_j}]$. Hence, the vector $(w_1, \ldots, w_t)^t$ is in the range of the matrix $P = [\langle v_j, v_i \rangle]$. Therefore, there are scalars $\alpha_1, \ldots, \alpha_t$ such that $\left\langle \sum_{j=1}^t \alpha_j v_j, v_i \right\rangle = w_i$ for $i = 1, \ldots, t$. Let $\xi = \sum_{j=1}^t \alpha_j v_j \in \ell^2$. We let ξ_i denote the ith component of ξ. Consider the function $F = (\xi_1 e, \xi_2 e, \ldots)^t$, which belongs to $C(\mathcal{A})$, because $\xi \in \ell^2$. We have that $F(x) = \xi$ if $x \in Y$ and is 0 if $x \in \{x_1, \ldots, x_n\} \setminus Y$.

From the argument in the previous paragraph we see that for each k such that $1 \leq k \leq p$ we can find ξ_k such that $\langle \xi_k, v_i \rangle = w_i$ for $n_{k-1} < i \leq n_k$. In addition, we can find F_k such that $F_k(x) = \xi_k$ for $x \in X_k$ and $F_k(x) = 0$ if $x \in \{x_1, \dots, x_n\} \setminus X_k$. Hence,

$$F_k(x)^* v_i = \begin{cases} \overline{w_i} & \text{if } x \in X_k \\ 0 & \text{if } x \in \{x_1, \dots, x_n\} \setminus X_k \end{cases}.$$

Hence, the function $F = F_1 + \cdots + F_p$ has the property that $F(x_i)^* v_i = w_i$ for $i = 1, \dots, n$.

Let $\mathcal{J}$ be the set of functions $F \in C(\mathcal{A})$ such that $F(x_j)^* v_j = 0$ for $j = 1, \dots, n$. We claim that $[\mathcal{J} g]$ is the set of functions in $\mathcal{H}_g \otimes \ell^2$ such that $\langle f(x_j), v_j \rangle = 0$.

One inclusion is straightforward. If $F_m g \to h$, then

$$\langle v_j, h(x_j) \rangle = \lim_{m \to \infty} \langle v_j, F_m(x_j) g(x_j) \rangle = \overline{g(x_j)} \lim_{m \to \infty} F_m(x_j)^* v_j = 0.$$

The reverse inclusion is a little more involved. Let $f \in [\mathcal{J} g]$. There exists a sequence $F_m \in C(\mathcal{A})$ such that $\|F_m g - f\| \to 0$. We need to modify F_m to a sequence $\tilde{F_m}$ such that $\left\|\tilde{F_m} g - f\right\| \to 0$ and $\tilde{F_m} \in \mathcal{J}$. Once again let $X_1, \dots, X_p$ be the partition of the set $\{x_1, \dots, x_n\}$ given by the equivalence relation of point separation.

Given a function $a \in \mathcal{A}$ we define $a \otimes \xi$ to be the function in $C(\mathcal{A})$ given by $(a \otimes \xi) h = ah \otimes \xi$. Let $\eta_{k,m}$ be the orthogonal projection of the vectors $F_m(x_{n_k})$ onto the finite-dimensional subspace spanned by $\{v_i : n_{k-1} < i \leq n_k\}$. Since, $\|F_m g - f\| \to 0$, we have that $\langle F_m(x_i) g(x_i) - f(x_i), v_i \rangle = \langle F_m(x_i), v_i \rangle \to 0$ for $i = 1, \dots, n$. Hence, $\|\eta_{k,m}\| \to 0$ as $m \to \infty$.

Let $G_m = \sum_{k=1}^p e_k \otimes \eta_{k,m}$, let $1 \leq i \leq n$, and let l be such that $n_{l-1} < i \leq n_l$. We have,

$$G_m(x_i)^* v_i = \sum_{k=1}^p (e_k \otimes \eta_{k,m})(x_i)^* v_i = \sum_{k=1}^p \overline{e_k(x_i)} \langle v_i, \eta_{k,m} \rangle.$$

Since $e_k|_{X_l}(x) = \delta_{k,l}$, the terms in the above sum for $k \neq l$ are zero. Hence, the sum reduces to $\langle v_i, \eta_{l,m} \rangle$. Recall that $\eta_{l,m}$ is the projection of $F(x_{n_l})$ onto the span of the vectors $\{v_i : n_{l-1} < i \leq n_l\}$. Hence, $\langle v_i, \eta_{l,m} \rangle = \langle v_i, F_m(x_{n_l}) \rangle = F_m(x_{n_l})^* v_i$. However, functions in $\mathcal{A}$ are constant on the sets X_k, which means $F_m(x_{n_l}) = F_m(x_i)$ and we get $G_m(x_i)^* v_i = F_m(x_i)^* v_i$ for all $i = 1, \dots, n$. Hence, the function $\tilde{F_m} = F_m - G_m \in \mathcal{J}$.

We have,

$$\|G_m g\| = \left\| \sum_{k=1}^p (e_k \otimes F_m(x_{n_k})) g \right\| \leq \sum_{k=1}^p \|e_k g \otimes \eta_{k,m}\| = \sum_{k=1}^p \|e_k g\| \, \|\eta_{k,m}\| \to 0$$

as $m \to \infty$.

Finally, note that $f \in \mathcal{H}_g \otimes \ell^2$ is orthogonal to $k_x \otimes v$ if and only if $\langle f(x), v \rangle = 0$. Hence, $(\mathcal{H}_g \otimes \ell^2) \ominus [\mathcal{J} g]$ is the span of the vectors $\{k_{x_i} \otimes v_i : i = 1, \dots, n\} = \mathcal{K}_g$. $\square$

The distance formula, equation (8), in Theorem 3.1 shows that we must be able to classify the cyclic subspaces of the form $(\mathcal{A} \otimes I)h$, and to do so, we begin with a simple lemma. We will use the natural identification between $\mathcal{H} \otimes \ell^2$ and the ℓ^2-direct sum of $\mathcal{H}$, which we denote $\oplus_{j=1}^\infty \mathcal{H}$.

Lemma 4.6. *Let $\mathcal{A} \subseteq B(\mathcal{H})$ and let $h \in \mathcal{H}$, then the cyclic subspace of $\mathcal{H}$ generated by $C(\mathcal{A})$ and h is equal to $[\mathcal{A}h] \otimes \ell^2$.*

Proof. The space $C(\mathcal{A})h$ is generated by elements of the form $ah \otimes e_j$ for $j \in \mathbb{N}$. Hence, $[\mathcal{A}h] \otimes \ell^2 \subseteq [C(\mathcal{A})h]$. On the other hand, an element of $C(\mathcal{A})h$ is of the form $\oplus_{j=1}^{\infty} a_j h$ where $\sum_{j=1}^{\infty} \|a_j h\|^2$ is finite and hence is in $[\mathcal{A}h] \otimes \ell^2$. □

4.3. **Proof of Theorem 1.1.** We will now prove our main theorem, Theorem 1.1, which is a tangential interpolation result for weak*-closed subalgebras of H^∞. Let $\mathcal{A} \subseteq H^\infty$ be a unital weak*-closed subalgebra of H^∞.

So far, we have assumed no additional structure on the algebras. A function $g \in H^2$ is called outer if and only if $H^\infty g$ is dense in H^2. When $\mathcal{A}$ is subalgebra of H^∞ the space $(C(\mathcal{A}) \otimes I)h$, which is contained in $(H^2 \otimes \ell^2) \otimes \ell^2$, can be identified with a subspace of $H^2 \otimes \ell^2$ of the form $C(\mathcal{A})g$ for some outer function g.

Lemma 4.7. *Let $\{h_i\}_{i=1}^{\infty}$ be a sequence in H^2 such that $\sum_{i=1}^{\infty} \|h_i\|^2$ is finite. Then the function $p(t) = \sum_{i=1}^{\infty} |h_i(t)|^2 \in L^1(\mathbb{T})$ and there exists an outer function $g \in H^2$ such that $p = |g|^2$ a.e. $\mathbb{T}$.*

Proof. The fact that $p \in L^1(\mathbb{T})$ is a straightforward argument.

It is well-known that a non-negative function $p \in L^1(\mathbb{T})$ is of the form $p = |g|^2$ for some outer function if and only if the function $\log p$ is summable. Let $p_m = \sum_{i=1}^{m} |h_i|^2$. If u_1 denotes the outer part of h_1, then $p_1 = |h_1|^2 = |u_1|^2$. Hence $\log p_1 \in L^1$. Since $\log p < p$ it follows, since $p \in L^1$, that $(\log p)^+ \in L^1$. On the other hand we have $\log p \geq \log p_1$ and so $(\log p)^- \leq (\log p_1)^-$. It follows that $(\log p)^- \in L^1$. □

Lemma 4.8. *Let $\oplus_{i=1}^{\infty} h_i \in H^2 \otimes \ell^2$ and let g be as in Lemma 4.7. If $\mathcal{A}$ is a unital subalgebra of H^∞, and $h = \sum_{j=1}^{\infty} h_j \otimes e_j$, then the map $U : [(C(\mathcal{A}) \otimes I)h] \to [C(\mathcal{A})g]$ defined by $U[((M_F \otimes I)h)] = [M_F g]$ is a unitary operator.*

Proof. The map is clearly linear and surjective. We now show that U is isometric, which also proves that U is well-defined.

We have,

$$\begin{aligned}
\|M_F g\|^2 &= \int \|Fg\|^2 = \int \sum_{i=1}^{\infty} |f_i g|^2 \\
&= \int \sum_{i=1}^{\infty} |f_i|^2 \left(\sum_{j=1}^{\infty} |h_j|^2\right) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \int |f_i h_j|^2 \\
&= \sum_{j=1}^{\infty} \|M_F h_j\|^2 = \|(M_F \otimes I)h\|^2 .
\end{aligned}$$

□

We are now in a position to prove our main result, which is a tangential interpolation theorem for subalgebras of H^∞. Given an outer function we will denote by $\mathcal{H}_g$ the closed subspace H^2 generated by elements of the form fg, where $f \in \mathcal{A}$. We will denote by K_g the kernel function for this subspace. When g is the constant function $g \equiv 1$ we will suppress the subscript g.

Proof of Theorem 1.1. We have seen that the existence of a solution F of norm at most α implies $Q_g \geq 0$ for all g. Hence, it is the converse that concerns us.

Let us assume that there is at least one solution, say F_0, which exists by Lemma 4.5. We view $\mathcal{A}$ as a subalgebra of $B(\mathcal{H})$. We define $\mathcal{J}$ to be the set of functions in $C(\mathcal{A})$ such that $F(x_j)^* v_j = 0$ for $j = 1, \ldots, n$.

Applying the distance formula (8), we get

$$d(F_0, \mathcal{J}) = \sup\{|\langle (M_{F_0} \otimes I)h, k\rangle|\},$$

where $h \in \mathcal{H}\otimes\ell^2$, $k \in (\mathcal{H}\otimes\ell^2)\otimes\ell^2$, $\|h\| = \|k\| = 1$ and $k \perp (\mathcal{J}\otimes I)h$. By projecting onto the subspace $[(C(\mathcal{A})\otimes I)h]$ we can assume that $k \in [(C(\mathcal{A})\otimes I)h]\ominus[(\mathcal{J}\otimes I)h]$. Let U be the unitary map from Lemma 4.8 and let g be the outer function such that $|g|^2 = \sum_{i=1}^{\infty} |h_i|^2$. We have,

$$\langle M_{F_0} h, k\rangle = \langle UM_{F_0} h, Uk\rangle = \langle M_{F_0} g, Uk\rangle = \langle M_{F_0} g, k'\rangle,$$

where $k' = Uk$. This gives

$$d(F_0, \mathcal{J}) = \sup\{|\langle M_{F_0} g, Uk\rangle|\} = \sup\{|\langle M_{F_0} g, k'\rangle|\},$$

where the supremum is over all outer functions $g \in H^2$ such that $\|g\| = 1$, $|g|^2 = \sum_{j=1}^{\infty} |h_i|^2$, $\|k'\| \leq 1$, and $k' \in [C(\mathcal{A})g] \ominus [\mathcal{J}g]$.

Lemma 4.5 shows that $[\mathcal{J}g]$ is the set of functions in $f \in \mathcal{H}_g \otimes \ell^2$ such that $\langle f(x_j), v_j\rangle = 0$ for $j = 1, \ldots, n$. Since $\langle f, k_x \otimes \xi\rangle = \langle f(x), \xi\rangle$, we see that $\mathcal{K}_g := [C(\mathcal{A})g] \ominus [\mathcal{J}g]$ is the span of the vectors $\{k^g_{x_i} \otimes v_i : i = 1, \ldots, n\}$, where k^g_x is the kernel function for $\mathcal{H}_g$ at the point x.

Therefore,

$$d(F_0, \mathcal{J}) \leq \sup\left|\langle g, M^*_{F_0} k'\rangle\right| \leq \sup\left\|M^*_{F_0}|_{\mathcal{K}_g}\right\|$$

where the supremum is taken over all outer functions g as above. If $F \in \mathcal{J}$, then $M^*_F(k^g_{x_i} \otimes v_i) = F(x_i)^* v_i k^g_{x_i} = 0$, and so $M^*_F|_{\mathcal{K}_g} = 0$. Hence, $\|F_0 + F\| \geq \left\|M^*_{F_0+F}|_{\mathcal{K}_g}\right\| = \left\|M^*_{F_0}|_{\mathcal{K}_g}\right\|$ from which it follows that $d(F_0, \mathcal{J}) = \sup_g \left\|M^*_{F_0}|_{\mathcal{K}_g}\right\|$.

The calculation leading to (5) shows that $\left\|M^*_{F_0}|_{\mathcal{K}_g}\right\| \leq \alpha$ if and only if the matrix $Q_g \geq 0$. Hence, the positivity of all the matrices Q_g implies that $d(F_0, \mathcal{J}) \leq \alpha$. This in turn guarantees the existence of a solution to the tangential problem of norm at most α. □

The proof of Corollary 1.2 is a consequence of the following observation.

If $h \in \mathcal{H}$, then there exists a sequence $f_n \in \mathcal{A}$ such that $\|f_n - h\|_2 \to 0$. Hence, $\left\||f_n|^2 - |h|^2\right\|_1 \leq \|f_n - h\|_2 \|f_n + h\|_2 \to 0$ as $n \to \infty$. Therefore $|h|^2 \in L^1(\mathcal{A})$. If $(h_n) \in \mathcal{H} \otimes \ell^2$ is a square summable sequence, then $\sum_{n=1}^{\infty} |h_n|^2 \in L^1(\mathcal{A})$. Hence, the absolute value of the outer function g such that $|g|^2 = \sum_{n=1}^{\infty} |h_n|^2$ is an element of $L^2(\mathcal{A})$.

We have now established a tangential interpolation theorem for weak*-closed subalgebras of H^∞. Proposition 4.2 shows that the tangential interpolation result implies a Töplitz corona theorem. This result can be viewed as analogous to the results obtained in Trent-Wick [16] and Douglas-Sarkar [10]. However, the method of proof is different.

4.4. **Examples.** A better feeling for the result in Theorem 1.1 can be obtained by examining some special cases. We single out two classes of examples of subalgebras of H^∞.

(1) Let B be an inner function and consider the algebra $H^\infty_B = \mathbb{C} + BH^\infty$. Note that $B \in H^\infty_B$ and so $\overline{B} \in L^\infty(H^\infty_B)$. Therefore, $H^\infty = \overline{B}BH^\infty$ is contained in $L^\infty(H^\infty_B)$ and we see that $L^\infty(H^\infty_B) = L^\infty$.

We can also give a more explicit description of the subspaces $\mathcal{H}_g$ in this case. Let g be an outer function and let $v = P_{H^2 \ominus BH^2} g$. We claim that $\mathcal{H}_g = [v] \oplus BH^2$. We have,

$$[(\mathbb{C} + BH^\infty)g] = \mathbb{C}g + B[H^\infty g] = \mathbb{C}v \oplus BH^2.$$

(2) Let R be finite open Riemann surface. It is well-known that the universal covering space for R is the open unit disk $\mathbb{D}$. Let $p : \mathbb{D} \to R$ denote the covering map and let Γ denote the set of deck transformations, that is, automorphisms γ of the disk such that $p \circ \gamma = p$.

The automorphisms in the group Γ act on the disk and induce an action on the space H^∞ by composition. The set of fixed points H^∞_Γ is naturally identified with the space of bounded holomorphic functions on the Riemann surface. The automorphisms also act by bounded linear maps on the space H^p and L^p and we use a subscript Γ to denote the associated set of fixed points.

In this case the algebra $L^\infty(H^\infty_\Gamma) = L^\infty_\Gamma$.

The cyclic subspace $\mathcal{H}_g$ for an outer function $|g| \in L^2_\Gamma$ can be described in terms of character automorphic function. A character of Γ is a homomorphism from Γ into the circle group $\mathbb{T}$ and we denote the space of characters by $\hat{\Gamma}$. A function $h \in H^2$ is called character automorphic if there exists a character $\sigma \in \hat{\Gamma}$ such that $h \circ \gamma = \sigma(\gamma)h$. The closure of H^∞_Γ in H^2 is the space H^2_Γ. If g is an outer function such that $|g| \in L^2(\mathcal{A})$, then there exists a character $\sigma \in \hat{\Gamma}$ such that $g \circ \gamma = \sigma(\gamma)g$. In addition, the space $\mathcal{H}_g = H^2_\sigma := \{f \in H^2 : f \circ \gamma = \sigma(\gamma)f\}$. A proof of these facts can be found in [14].

Given a character σ we let K^σ denote the reproducing kernel of H^2_σ. We get that the kernels of the character automorphic spaces H^2_σ, where $\sigma \in \hat{\Gamma}$ have the tangential interpolation property for H^∞_Γ.

In Section 5 we will return to this example and show that we can replace this family of matrix positivity conditions by a single condition, at the expense of the optimal constant for the norm of a solution to the tangential interpolation problem.

We point out that the tangential interpolation theorem gives us a new way to derive the Nevanlinna-Pick type interpolation results in [13, 14] for the examples above.

5. Applications of Theorem 1.3: Similar Cyclic Modules

Theorem 1.1 shows that the positivity of $M_F M_F^* \geq \delta^2$ on a family of reproducing kernel Hilbert spaces is enough to guarantee the existence of a function G such that $FG = 1$ and $\|M_G\| \leq \delta^{-1}$. This theorem is analogous to the results obtained in the work of [4, 10, 16].

If we drop the requirement that the function G have optimal norm, then in some cases we can replace the family of conditions by a single condition. Let $\mathcal{I}_x$ denote the ideal of functions in $\mathcal{A}$ such that $f(x) = 0$. We have already seen that $[\mathcal{I}_x g]$ is

a codimension one subspace of $\mathcal{H}_g$ and that the orthogonal complement of $\mathcal{I}_x g$ is spanned by the kernel function k_x^g.

Now let us return to the setting where $\mathcal{A}$ is a unital weak*-closed subalgebra of H^∞ and the function g is outer. We will establish a tangential interpolation theorem where we replace the family of conditions $Q_g \geq 0$ for all outer functions g such that $|g| \in L^2(\mathcal{A})$ by a single positivity condition. However, we can not guarantee a solution of optimal norm. We will then apply our result to the case of finite open Riemann surfaces.

Let g, h be two outer functions and let $S : \mathcal{H}_g \to \mathcal{H}_h$ be a bounded invertible operator that intertwines the action of $\mathcal{A}$, that is, such that $SM_f = M_f S$ for all $f \in \mathcal{A}$. By taking adjoints we see that $M_f^* S^* = S^* M_f^*$ for all $f \in \mathcal{A}$. If $x \in \mathbb{D}$, then $M_f^* k_x^g = \overline{f(x)} k_x^g$ and so we have $M_f^* S^* k_x^h = \overline{f(x)} S^* k_x^h$ for all $x \in \mathbb{D}$ and $f \in \mathcal{A}$. It follows that the vector $S^* k_x^h$ is orthogonal to $\mathcal{I}_x g$. Hence, $S^* k_x^h = \overline{\phi(x)} k_x^g$ for all $x \in \mathbb{D}$, where ϕ is a complex-valued function on the disk. In fact, ϕ is a multiplier from $\mathcal{H}_g \to H_h$ and $S = M_\phi$.

Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{K}_1$ and $\mathcal{K}_2$ be n-dimensional Hilbert spaces. If $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $x_1, \ldots, x_n$ is a basis for the space $\mathcal{H}_1$, then $\|A\| \leq \alpha$ if and only if the matrix $[\alpha^2 \langle x_j, x_i \rangle - \langle Ax_j, Ax_i \rangle]$ is positive semidefinite. Now let $S_i \in B(\mathcal{H}_i, \mathcal{K}_i)$ be a bounded invertible transformation. If $\|A\| \leq \alpha$, then $\left\|S_2 A S_1^{-1}\right\| \leq \alpha \left\|S_2\right\| \left\|S_1^{-1}\right\|$.

Consider the special case where $\mathcal{H}_1 = \operatorname{span}\{k_{x_1}^g \otimes v_1, \ldots, k_{x_n}^g \otimes v_n\}$, $\mathcal{K}_1 = \operatorname{span}\{k_{x_1}^h \otimes v_1, \ldots, k_{x_n}^h \otimes v_n\}$, $\mathcal{H}_2 = \operatorname{span}\{k_{x_1}^g, \ldots, k_{x_n}^g\}$, $\mathcal{K}_2 = \operatorname{span}\{k_{x_1}^h, \ldots, k_{x_n}^h\}$. Let A be the map $A(k_{x_i}^g \otimes v_i) = \overline{w_i} k_{x_i}^g$, let $S = M_\phi$ be the similarity between $\mathcal{H}_g$ and $\mathcal{H}_h$ described above, let $S_1 = S^* \otimes I$, and let $S_2 = S^*$.

Note that if $F \in C(\mathcal{A})$, with $F(x_i)^* v_i = \overline{w_i}$, then $M_F^*|_{\mathcal{H}_1}$ is precisely A.

If $\|S\| \left\|S^{-1}\right\| = c$, then a straightforward computation shows that $[(\alpha^2 \langle v_j, v_i \rangle - w_i \overline{w_j}) K_g(x_i, x_j)] \geq 0$ if and only if $\|A\| \leq \alpha$. This implies $\left\|S^* A((S^*)^{-1} \otimes I)\right\| \leq c\alpha$ which in turn implies that $[(c^2 \alpha^2 \langle v_j, v_i \rangle - w_i \overline{w_j}) K_h(x_i, x_j)] \geq 0$.

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. From the observations made above we see that the matrix positivity condition implies that $[(\alpha^2 c^2 \langle v_j, v_i \rangle - w_i \overline{w_j}) K^g(x_i, x_j)] \geq 0$. The result in (1) now follows from Theorem 1.1.

The proof of (2) follows, as before, from the tangential interpolation theorem established in (1). ☐

We now provide an example of a class of subalgebras of H^∞ to which the above theorem applies. Recall that if R is a finite open Riemann surface, and Γ is the associated group of deck transformations acting on the disk, then the fixed-point algebra H_Γ^∞ is naturally identified with $H^\infty(R)$.

In this case the outer function g has the property that there is a character σ such that $g \in H_\sigma^2$, and $\mathcal{H}_g = H_\sigma^2$. We will establish the existence of a similarity $S_\sigma = M_{\phi_\sigma}$ between H_Γ^2 and H_σ^2 and show that there is a uniform bound on $\|S_\sigma\| \|S_\sigma^{-1}\|$. This result generalizes a theorem of Ball [7] from the setting of multiply-connected domains to Riemann surfaces.

Proposition 5.1. *Let Γ be a the group of deck transformations associated to a finite open Riemann surface. For each $\sigma \in \hat{\Gamma}$, there exists a bounded invertible function ϕ_σ such that $\phi_\sigma H_\Gamma^2 = H_\sigma^2$. There is a constant β, independent of σ such that $\beta^{-1} \leq |\phi_\sigma| \leq \beta$.*

We will need two results of Forelli [11].

Theorem 5.2 (Forelli). *Let Γ be the group of deck transformations associated to a finite open Riemann surface R of genus m. Let $\gamma_1,\dots,\gamma_m$ denote the generators of Γ. There exist m vectors $v_1,\dots,v_m\in L^\infty_\Gamma$ such that v_i is non-negative, and v_i is orthogonal to $H^2_\Gamma\oplus\overline{H^2_{\Gamma,0}}$. In addition, $v_1,\dots,v_m$ are linearly independent.*

If f is a real-valued function in L^2, then its conjugate function f^* is the unique real-valued function in L^2 such that $f+if^*\in H^2$ and $\int f^*=0$.

Theorem 5.3 (Forelli). *Let $f\in L^2_\Gamma$ and let f^* denote the function conjugate to f, then $f^*\circ\gamma_i-f$ is constant, and the constant is given by $\int fv_i$.*

Now we present the proof of Proposition 5.1.

Proof of Proposition 5.1. Let $\gamma_1,\dots,\gamma_m$ be a minimal set of generators of the group Γ. We let $\sigma_k=\sigma(\gamma_k)$. Since σ is a character, there exists $\theta_1,\dots,\theta_m\in[0,2\pi)$ such that $\sigma_k=e^{i\theta_k}$.

Let $v_{k,l}=\langle v_l,v_k\rangle$ for $k,l=1,\dots,m$. Note that $v_l^*\circ\gamma_k=v_l^*+\int v_kv_l=v_l^*+\langle v_k,v_l\rangle$. Since $v_1,\dots,v_m$ are linearly independent the matrix $V=[v_{k,l}]$ is invertible. Let $c=(c_1,\dots,c_m)^t$ be the unique vector such that $Vc=(\theta_1,\dots,\theta_m)^t$. Since the entries of V are real we see that V^{-1} has real entries. Hence, the vector $c\in\mathbb{R}^m$.

Let $f=\sum_{k=1}^m c_kv_k$. Note that f is a real-valued element of L^∞. Following the construction in [11] we let $\phi_\sigma=\exp(f+if^*)$. Since f is real-valued we see that $|\phi_\sigma|=\exp(f)$ and so ϕ_σ is bounded. Now,

$$\begin{aligned}\phi_\sigma\circ\gamma_k&=\exp\left(f+if^*+i\sum_{l=1}^m c_lv_{k,l}\right)\\&=\exp\left(i\sum_{l=1}^m c_lv_{k,l}\right)\phi_\sigma=\exp(i\theta_k)\phi_\sigma=\sigma(\gamma_k)\phi_\sigma.\end{aligned}$$

Hence, $\phi_\sigma\in H^\infty_\sigma$.

We have $|\sum_{k=1}^m c_kv_k|\le\max_{k=1,\dots,m}\|v_k\|_\infty\|c\|_1$. Since $\theta_1,\dots,\theta_m\in[0,2\pi)$ there exists a constant K, that does not depend on σ, such that $\|c\|_1\le K$. Hence, there is a constant K' such that $e^{-K'}\le|\phi_\sigma|\le e^{K'}$ for all $\sigma\in\hat\Gamma$. $\square$

References

[1] Jim Agler and John E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, vol. 44, American Mathematical Society, Providence, RI, 2002. ↑4

[2] ———, *Nevanlinna-Pick interpolation on the bidisk*, J. Reine Angew. Math. **506** (1999), 191–204. ↑3

[3] ———, *What Hilbert spaces can tell us about bounded functions in the bidisk*, available at http://www.arxiv.org/0901.0907. ↑3

[4] E. Amar, *On the Toëplitz corona problem*, Publ. Mat. **47** (2003), no. 2, 489–496. ↑14

[5] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404. ↑4

[6] William Arveson, *Interpolation problems in nest algebras*, J. Functional Analysis **20** (1975), no. 3, 208–233. ↑1, 6

[7] Joseph A. Ball, *Interpolation problems and Toeplitz operators on multiply connected domains*, Integral Equations Operator Theory **4** (1981), no. 2, 172–184. ↑3, 15

[8] Joseph A. Ball and Tavan T. Trent, *Unitary colligations, reproducing kernel Hilbert spaces, and Nevanlinna-Pick interpolation in several variables*, J. Funct. Anal. **157** (1998), no. 1, 1–61. ↑3

[9] Lennart Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math. (2) **76** (1962), 547–559. ↑1
[10] Ronald G. Douglas and Jaydeb Sarkar, *Some Remarks on the Toeplitz Corona problem*, Proceedings of CRM (2009). ↑1, 13, 14
[11] Frank Forelli, *Bounded holomorphic functions and projections*, Illinois J. Math. **10** (1966), 367–380. ↑16
[12] Scott McCullough, *Nevanlinna-Pick type interpolation in a dual algebra*, J. Funct. Anal. **135** (1996), no. 1, 93–131. ↑1, 6
[13] Mrinal Raghupathi, *Nevanlinna-Pick interpolation for* $\mathbb{C} + BH^\infty$, Integral Equations Operator Theory **63** (2009), no. 1. ↑6, 14
[14] ———, *Abrahamse's interpolation theorem and Fuchsian groups*, J. Math. Anal. Appl. **355** (2009), no. 1, 258–276. ↑14
[15] C. F. Schubert, *The corona theorem as an operator theorem*, Proc. Amer. Math. Soc. **69** (1978), no. 1, 73–76. ↑1
[16] Tavan T. Trent and Brett D. Wick, *Toeplitz Corona Theorems for the Polydisk and the Unit Ball*, Complex Anal. Oper. Theory **3** (2009), no. 3, 729–738. ↑1, 13, 14

Department of Mathematics, Vanderbilt University, Nashville, Tennessee, 37240 USA

E-mail address: `mrinal.raghupathi@vanderbilt.edu`
URL: `http://www.math.vanderbilt.edu/~mrinalr`

School of Mathematics, Georgia Institute of Technology, 686 Cherry Street, Atlanta, GA 30332-1060 USA

E-mail address: `wick@math.gatech.edu`
URL: `http://people.math.gatech.edu/~bwick6/`